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A FIXED POINT THEOREM FOR A CLASS OF STAR-SHAPED SETS IN c_0 †

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ABSTRACT

A subset K of c_0 is coordinatewise star-shaped (c.s.s.) if there exists a center point $x \in K$ such that for $y \in K$ and $z \in c_0$, if z is coordinatewise between x and y then $z \in K$. We prove that a weakly compact c.s.s. subset of c_0 has the fixed point property for nonexpansive mappings and that a fixed point for such a mapping can be obtained in a constructive manner.

1. Introduction

Let K be a closed subset of a Banach space and let $T: K \to K$ be nonexpansive ($\| Tx - Ty \| \leqq \| x - y \|$ for $x, y \in K$). It is still an open problem to give general conditions on K so that T must have a fixed point. Recently it has been shown by D. Alspach [1] that T may fail to have a fixed point if K is a convex weakly compact subset of $L_1(0,1)$. In [3] it was proved that if K is a subset of c_0 which is the closed convex hull of a weakly convergent sequence then T must have a fixed point. The proof of this fact was somewhat lengthy and technical.

In this paper we study the fixed point property for a different class of weakly compact (not necessarily convex) subsets of c_0 which we call coordinatewise star-shaped sets. For $x \in c_0$ we write $x = (x(i))$ if $x(i)$ is the i-th coordinate of x. c_0 is the Banach space of all sequences x of reals which converge to 0 with

$$\| x \| = \max\{ | x(i) | : 1 \leqq i < \infty \}.$$

Definition. A subset K of c_0 is said to be *coordinatewise star-shaped* (c.s.s.) if there exists a point $x \in K$ (called the center of K) such that for all $y \in K$ and $z \in c_0$, if $z(i) \in \operatorname{conv}\{x(i), y(i)\}$ for all i, then $z \in K$. Note that $\operatorname{conv}\{a, b\}$ is just the closed interval between a and b.

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Clearly the property of being c.s.s. is translation invariant, and thus we shall restrict ourselves to the case where the center point is 0. In this case K is c.s.s. if and only if for each $y \in K$ and $0 \leqq \alpha_i \leqq 1$ $(1 \leqq i < \infty)$ the vector $(\alpha_i y(i))_{i=1}^{\infty} \in K$.

Of course, c.s.s. sets may fail to be convex. One such set is the image K_p under the formal identity map of the unit ball of l_p $(0 < p < 1)$,

$$K_p = \left\{ x \in c_0 : \sum_{i=1}^{\infty} |x(i)|^p \leqq 1 \right\}.$$

Let us mention two other classes of sets closely related to c.s.s. A subset K of a linear space is called *star-shaped* with center x if for all $y \in K$ and $0 \leqq t \leqq 1$, $tx + (1-t)y \in K$. A subset K of a vector lattice X is called *solid* if $y \in K$, $z \in X$ and $|y| \geqq |z|$ imply $z \in K$. Clearly in c_0 a solid set is c.s.s. with center 0 and a c.s.s. set is star-shaped. Both converse implications are false.

The main result (section 2) of this paper is that weakly compact c.s.s. sets in c_0 have the fixed point property for nonexpansive mappings and that a fixed point of such a map can be obtained constructively. The proof is fairly easy. In order to explain what we mean by "constructively" we first set some notation and recall some easy facts.

Let K be a closed star-shaped set (with center 0) in a Banach space and let $T: K \to K$ be nonexpansive. Then for each $0 < t < 1$ the map tT given by $(tT)x = t(Tx)$ maps K into K and satisfies $\|(tT)x - (tT)y\| \leqq t\|x - y\|$ for all $x, y \in K$. Hence by the Banach contraction principle, tT has a unique fixed point y_t in K and in fact $y_t = \lim_{n\to\infty}(tT)^n x$ for any $x \in K$. Since $Ty_t = (1/t)y_t$, if for some sequence $t_n \uparrow 1$, $(y_{t_n})_{n=1}^{\infty}$ converges (in norm) to y, then $Ty = y$. However, $(y_{t_n})_{n=1}^{\infty}$ may fail to converge for any $t_n \uparrow 1$. Such an example is given by the well known self map T of the closed unit ball of c_0 defined by

$$Tx = (1 - \|x\|, x(1), x(2), \cdots).$$

In this case, $y_t = (1/1+t)(t, t^2, t^3, \cdots)$ and T has no fixed point.

Definition. Let K be a closed star-shaped subset of a Banach space. K is said to have the *effective fixed point property* for nonexpansive mappings if for each such map $T: K \to K$ and each sequence $(t_n)_{n=1}^{\infty}$ increasing to 1, there exists a subsequence (t'_n) so that $(y_{t'_n})_{n=1}^{\infty}$ is norm convergent.

2. The main result

Theorem 1. *Weakly compact, coordinatewise star-shaped subsets of c_0 have the effective fixed point property for nonexpansive mappings.*

COROLLARY. *Every weakly compact subset of c_0 is contained in a set with the effective fixed point property for nonexpansive mappings.*

The corollary follows immediately from Theorem 1 since every weakly compact subset K of c_0 is contained in a solid weakly compact set $S(K)$ in c_0. It would be interesting to determine when there is a nonexpansive retract from $S(K)$ onto K for this would imply K also has the fixed point property. For some interesting work on nonexpansive retracts see [2]. The rest of this section is devoted to the proof of Theorem 1.

Let K be a c.s.s. subset of c_0 with center 0. Define an order relation on c_0 by $y \prec x$ if for all $i = 1, 2, \cdots, y(i) \in \operatorname{conv}\{0, x(i)\}$. Thus if $x \in K$, $y \in c_0$ and $y \prec x$ then $y \in K$.

DEFINITION. For $x \in c_0$ and $r > 0$ let $x \circ r \in c_0$ be given by

$$(x \circ r)(i) = \begin{cases} 0 & \text{if } |x(i)| \leqq r, \\ x(i) - r & \text{if } x(i) > r, \\ x(i) + r & \text{if } x(i) < -r. \end{cases}$$

We begin with two very simple observations.

PROPOSITION 1. *If $x, y \in c_0$, $r > 0$ and $\|x - y\| \leqq r$ then $y \circ r \prec x$.*

PROOF. We have that for all i, $y(i) - r \leqq x(i) \leqq y(i) + r$. If $y(i) > r$ then $0 < (y \circ r)(i) = y(i) - r \leqq x(i)$ and if $y(i) < -r$ then $0 > (y \circ r)(i) = y(i) + r \geqq x(i)$. If $|y(i)| \leqq r$ then $(y \circ r)(i) = 0$. Thus $y \prec x$. □

PROPOSITION 2. *Let r and s be nonnegative numbers with $1 < s < 1 + r$ and let $y \in c_0$ with $\|y\| \leqq 1$. Then*

$$\|y - [(sy) \circ r]\| \leqq r/s.$$

PROOF. Fix a coordinate i and consider the following three cases: (i) $|sy(i)| \leqq r$, (ii) $sy(i) > r$ and (iii) $sy(i) < -r$. In case (i), $(sy \circ r)(i) = 0$ and so $|y(i) - (sy \circ r)(i)| = |y(i)| \leqq r/s$. In case (ii), $(sy \circ r)(i) = sy(i) - r$ and so

$$|y(i) - (sy \circ r)(i)| = |y(i) - (sy(i) - r)| = |r - (s-1)y(i)|$$

$$= r - (s-1)y(i) \leqq r - (s-1)r/s = r/s.$$

The third equality holds since $s - 1 < r$. A similar calculation applies to case (iii).

□

We may assume our weakly compact c.s.s. K with center 0 has diameter bounded by 1. Let $T: K \to K$ be nonexpansive and let $t_i \uparrow 1$. By the weak compactness of K there exists a subsequence of (t_l) which we still denote by (t_l) such that $(y_{t_l})_{l=1}^{\infty}$ converges weakly to some element $y_0 \in K$. We shall eventually prove that in fact $\lim_{l\to\infty} \| y_{t_l} - y_0 \| = 0$. If not, then there exists $r > 0$ and a further subsequence which we still call (y_{t_l}) such that

$$\lim_{l\to\infty} \| y_{t_l} - y_0 \| = r.$$

The remainder of the argument will be to show that this violates the weak compactness of K.

Let $(\alpha_n)_{n=0}^{\infty}$ be a sequence of reals rapidly decreasing to 0 ($\alpha_n = r2^{-(n+10)}$ will do nicely). We inductively choose a sequence $(i_n)_{n=0}^{\infty}$ of positive integers and a subsequence $(t'_n)_{n=1}^{\infty}$ of $(t_l)_{l=1}^{\infty}$ so that if $y_n = y_{t'_n}$, then

$$(1) \qquad \big| \| y_n - y_0 \| - r \big| < \alpha_n \qquad (n = 1, 2, \cdots),$$

$$(2) \qquad s_n = 1/t'_n < 1 + \alpha_n \qquad (n = 1, 2, \cdots),$$

$$(3) \qquad | y_k(i) | < \alpha_n \qquad (i \geqq i_n, 0 \leqq k \leqq n, n = 1, 2, \cdots),$$

$$(4) \qquad | y_n(i) - y_0(i) | < \alpha_n \qquad (1 \leqq i \leqq i_{n-1}, n = 1, 2, \cdots)$$

are true. Note that by (2) we now have $Ty_n = s_n y_n$ for all n.

Indeed choose $i_0 > 1$ so that $| y_0(i) | < \alpha_0$ for $i \geqq i_0$. Choose t'_1 so that (1), (2) and (4) hold for $n = 1$. (Recall that $(y_{t_l})_{l=1}^{\infty}$ converges weakly to y_0 so for each i, $\lim_{l\to\infty} y_{t_l}(i) = y_0(i)$.) Choose $i_1 > i_0$ so that for $i \geqq i_1$, $| y_0(i) | < \alpha_1$ and $| y_1(i) | < \alpha_1$. Let $t'_2 > t'_1$ be such that (1), (2) and (4) hold for $n = 2$ and then choose $i_2 > i_1$ so that (3) holds for $n = 2$. Continue in this manner.

Let $F_0 = \{ i : 1 \leqq i < i_0 \}$ and $F_n = \{ i : i_{n-1} \leqq i < i_n \}$ for $n = 1, 2, \cdots$. The vector y_n is "essentially supported" on $F_0 \cup F_n$ and the norm of $y_n - y_0$ is attained at some coordinate of F_n.

We inductively construct a sequence $(x_k)_{k=0}^{\infty}$ in K which fails to have a weakly convergent subsequence. Set

$$(5) \qquad x_0 = y_0 |_{F_0} \qquad \text{i.e. } x_0(i) = \begin{cases} 0, & i \notin F_0, \\ y_0(i), & i \in F_0, \end{cases}$$

$$(6) \qquad r_n^0 = \| x_0 - y_0 \| \qquad (n = 1, 2, \cdots),$$

$$(7) \qquad x_{k+1} = \bigvee_{n=1}^{\infty} (s_n y_n \circ r_n^k) \qquad (k = 0, 1, 2, \cdots),$$

$$(8) \qquad r_n^{k+1} = \max\{ \| x_{k+1} - y_n \|, 7r/8 \} \quad (k = 0, 1, 2, \cdots; n = 1, 2, \cdots).$$

The "sup" in (7) is taken with respect to the order $<$. Thus $\bigvee_n w_n$ is defined only if for each i, sign $w_n(i) =$ sign $w_m(i)$ for all n and m and then

$$\bigvee_n w_n(i) = \begin{cases} \sup_n w_n(i), & \text{if all } w_n(i) \geqq 0, \\ \inf_n w_n(i), & \text{if all } w_n(i) < 0. \end{cases}$$

We check that x_k is well defined and $x_k \in K$ for each k. First note $x_0 < y_0$ and so $x_0 \in K$. Assume that $x_0, x_1, \cdots, x_k$ are all well defined and belong to K and $r_n^j = \max\{\|x_j - y_n\|, 7r/8\}$ $(0 \leqq j \leqq k;\quad n = 1, 2, \cdots)$. Then in particular, $\|x_k - y_n\| \leqq r_n^k$ and so $\|Tx_k - s_n y_n\| = \|Tx_k - Ty_n\| \leqq \|x_k - y_n\| \leqq r_n^k$. By Proposition 1, $Tx_k > (s_n y_n \circ r_n^k)$ for all n. Thus $x_{k+1} = \bigvee_n (s_n y_n \circ r_n^k) < Tx_k$ is well defined and $x_{k+1} \in K$.

The proof of Theorem 1 will be complete if we show (x_k) has no weakly convergent subsequence. Let

(9) $$\varepsilon_n = 7r(s_n - 1)/8s_n > 0.$$

We shall prove that

(10) $$\|x_{k+1} - y_n\| \leqq r_n^k - \varepsilon_n \qquad \text{for all } n \text{ and } k.$$

Assume for the moment that (10) is true. Fix $n \geqq 1$. Then $r_n^{k+1} = \max\{\|x_{k+1} - y_n\|, 7r/8\} \leqq \max\{r_n^k - \varepsilon_n, 7r/8\}$. Since ε_n is a positive number depending solely upon n, there exists $k(n)$ so that for $k \geqq k(n)$, $r_n^k = 7r/8$. Choose $j_n \in F_n$ with $\|y_n - y_0\| = |y_n(j_n) - y_0)|$. Then by (1), $|y_n(j_n) - y_0(j_n)| > r - \alpha_n$ and so by (3), $|y_n(j_n)| \geqq r - \alpha_n - |y_0(j_n)| > r - 2\alpha_n > 8r/9$. Suppose $k \geqq k(n)$. Then $\|x_k - y_n\| \leqq r_n^k = 7r/8$ and so $|x_k(j_n)| \geqq 8r/9 - 7r/8 = r/72$. Since the coordinates (j_n) are distinct it follows that (x_k) has no weakly convergent subsequence in c_0. Thus it remains only to prove (10).

Fix n and k. By definition $r_n^k \geqq 7r/8$, and by (2), $1 < s_n < 1 + \alpha_n < 1 + 7r/8 \leqq 1 + r_n^k$. Thus by Proposition 2,

(11) $$\|y_n - (s_n y_n \circ r_n^k)\| \leqq r_n^k/s_n = r_n^k - (s_n - 1)r_n^k/s_n < r_n^k - 7(s_n - 1)r/8s_n = r_n^k - \varepsilon_n.$$

So to prove (10) it suffices to show

(12) $$|y_n(i) - x_{k+1}(i)| \leqq \max\{|y_n(i) - (s_n y_n \circ r_n^k)(i)|, r_n^k - \varepsilon_n\} \qquad \text{for } i = 1, 2, \cdots.$$

Of course, by (11), the "max" in (12) is just $r_n^k - \varepsilon_n$, but we choose to state our problem in this form for technical reasons. Also, it is actually true that $\| y_n - x_{k+1} \| \leqq \| y_n - (s_n y_n \circ r_n^k) \|$.

Fix a coordinate i. Now $x_{k+1}|_{F_n} = (s_n y_n \circ r_n^k)|_{F_n}$ by the definition of x_{k+1} and the fact that $|y_m(i)| < r/2$ if $i \in F_n$, $m \neq n$. Thus (11) implies (12) for the case $i \in F_n$.

Suppose $i \in F_m$ with $1 \leqq m < \infty$, $m \neq n$. Then by (1), (3) and (4), $|y_n(i)| \leqq |y_n(i) - y_0(i)| + |y_0(i)| < \alpha_n + \alpha_m < 2\alpha_0$, and $|y_m(i)| \leqq |y_m(i) - y_0(i)| + |y_0(i)| \leqq r + \alpha_m + \alpha_m < r + 2\alpha_0$. Hence if $x_{k+1}(i) > 0$ (the case $x_{k+1}(i) < 0$ is handled similarly), then $x_{k+1}(i) = s_m y_m \circ r_m^k(i) = s_m y_m(i) - r_m^k \leqq s_m(r + 2\alpha_0) - 7r/8 = (s_m - 7/8)r + 2s_m\alpha_0 \leqq r/4 + 3\alpha_0$. Thus $|y_n(i) - x_{k+1}(i)| \leqq 2\alpha_0 + r/4 + 3\alpha_0 < r/2$, and $r_n^k - \varepsilon_n > 7r/8 - 7(s_n - 1)r/8s_n > 7r/8 - 7\alpha_n r/8 > r/2$.

Lastly suppose $i \in F_0$. We shall assume $y_n(i) \geqq 0$, the argument for $y_n(i) < 0$ being similar. There are two cases.

Case 1. $s_n y_n(i) \leqq r_n^k$.

Then $(s_n y_n \circ r_n^k)(i) = 0$ and we need only show that $|y_n(i) - x_{k+1}(i)| \leqq y_n(i)$ or

$$0 \leqq x_{k+1}(i) \leqq 2y_n(i). \tag{13}$$

Now for $n, m \geqq 1$, by (4),

$$|y_n(i) - y_m(i)| \leqq |y_n(i) - y_0(i)| + |y_0(i) - y_m(i)| \leqq \alpha_n + \alpha_m \leqq \alpha_0.$$

Thus $y_n(i) - \alpha_0 \leqq y_m(i) \leqq y_n(i) + \alpha_0$. Since $y_n(i) \geqq 0$, $y_m(i) \geqq -\alpha_0$ for all m and $s_m y_m(i) \geqq -s_m\alpha_0 \geqq -7r/8 > -r_m^k$. Thus $(s_m y_m \circ r_m^k)(i) \geqq 0$ for all m, whence $x_{k+1}(i) \geqq 0$ which proves one half of (13).

Also, for all m

$$s_m y_m(i) \leqq s_m y_n(i) + s_m\alpha_0 < 2y_n(i) + r/2 < 2y_n(i) + r_m^k.$$

Thus $(s_m y_m \circ r_m^k)(i) \leqq \max\{0, s_m y_m(i) - r_m^k\} \leqq 2y_n(i)$. This proves (13).

Case 2. $s_n y_n(i) > r_n^k$.

In this case, $0 < (s_n y_n \circ r_n^k)(i) \leqq x_{k+1}(i)$. For all m, $0 \leqq (s_m y_m \circ r_m^k)(i) \leqq \max\{0, s_m y_m(i) - r_m^k\} \leqq \max\{0, s_m(y_n(i) + \alpha_0) - r_m^k\} \leqq y_n(i)$ (since $s_m y_n(i) - y_n(i) + s_m\alpha_0 = (s_m - 1)y_n(i) + s_m\alpha_0 < 7r/8 \leqq r_m^k$). Thus $(s_n y_n \circ r_n^k)(i) \leqq x_{k+1}(i) \leqq y_n(i)$ and so

$$|y_n(i) - x_{k+1}(i)| = y_n(i) - x_{k+1}(i) \leqq y_n(i) - (s_n y_n \circ r_n^k)(i) = |y_n(i) - (s_n y_n \circ r_n^k)(i)|.$$

Thus (12) holds in this case as well. □

REFERENCES

1. D. E. Alspach, *A fixed point free nonexpansive map*, preprint.
2. D. G. de Figueiredo and L. Karlovitz, *On the extension of contractions on normed spaces*, in *Nonlinear Functional Analysis*, Proc. Symp. Pure Math., Vol XVIII, Part 1, 1968, pp. 95–104.
3. E. Odell and Y. Sternfeld, *A fixed point theorem in* c_0, to appear.

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